

A NON-VARIATIONAL SYSTEM INVOLVING THE CRITICAL SOBOLEV EXPONENT. THE RADIAL CASE.

FRANCESCA GLADIALI, MASSIMO GROSSI, AND CHRISTOPHE TROESTLER

ABSTRACT. In this paper we consider the non-variational system

$$\begin{cases} -\Delta u_i = \sum_{j=1}^k a_{ij} u_j^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ u_i > 0 & \text{in } \mathbb{R}^N, \\ u_i \in D^{1,2}(\mathbb{R}^N). \end{cases} \quad (0.1)$$

and we give some sufficient conditions on the matrix $(a_{ij})_{i,j=1,\dots,k}$ which ensure the existence of solution bifurcating from the bubble of the critical Sobolev equation.

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2010 *Mathematics Subject Classification.* 35B09, 35B32, 35B33, 35J47.

Key words and phrases. Elliptic system, positive solutions, critical exponent, bifurcation, spectrum at the standard bubble, Crandall-Rabinowitz theorem, Pohozaev identity.

The first author is supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The first two authors are supported by PRIN-2012-grant "Variational and perturbative aspects of nonlinear differential problems". The third author is partially supported by the project "Existence and asymptotic behavior of solutions to systems of semilinear elliptic partial differential equations" (T.1110.14) of the *Fonds de la Recherche Fondamentale Collective*, Belgium.

1. INTRODUCTION

1.1. Setting of the problem. In this paper we consider the $k \times k$ system

$$\begin{cases} -\Delta u_i = \sum_{j=1}^k a_{ij} u_j^{2^*-1} & \text{in } \mathbb{R}^N, \\ u_i > 0 & \text{in } \mathbb{R}^N, \\ u_i \in D^{1,2}(\mathbb{R}^N), \end{cases} \quad (1.1)$$

for $i = 1, \dots, k$, where $N \geq 3$, $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \text{ such that } |\nabla u| \in L^2(\mathbb{R}^N)\}$, $2^* = \frac{2N}{N-2}$ and the matrix $A := (a_{ij})_{i,j=1,\dots,k}$ satisfies

$$A \text{ is symmetric}, \quad (1.2)$$

$$\sum_{j=1}^k a_{ij} = 1 \quad \text{for any } i = 1, \dots, k. \quad (1.3)$$

Assumptions (1.2) and (1.3) imply that A possesses $(k-1)k/2$ free parameters, i.e., the set of matrices satisfying (1.2)–(1.3) form an affine subspace of dimension $(k-1)k/2$. For example, for $k = 3$, such a matrix can be written

$$A = \begin{pmatrix} 1 - \alpha_1 - \alpha_2 & \alpha_1 & \alpha_2 \\ \alpha_1 & 1 - \alpha_1 - \alpha_3 & \alpha_3 \\ \alpha_2 & \alpha_3 & 1 - \alpha_2 - \alpha_3 \end{pmatrix} \quad \text{for } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.$$

If $u_1 = u_2 = \dots = u_k$ then (1.1) reduces to the classical critical Sobolev equation

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \quad (1.4)$$

Actually, our problem can be seen as a straightforward generalization of Equation (1.4) to the case of systems.

It is well known (see [CGS]) that (1.4) admits the $(N+1)$ -parameter family of solutions given by the standard bubbles

$$U_{\delta,y}(x) := \frac{[N(N-2)\delta^2]^{\frac{N-2}{4}}}{(\delta^2 + |x-y|^2)^{\frac{N-2}{2}}}$$

where $\delta > 0$ and $y \in \mathbb{R}^N$. For simplicity we will denote by

$$U(x) := U_{1,0}(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1 + |x|^2)^{\frac{N-2}{2}}}. \quad (1.5)$$

In this paper we want to prove the existence of solutions to Problem (1.1) that are different from the *trivial* solution $(U_{\delta,y}, \dots, U_{\delta,y})$.

We now show how our system, choosing particular matrices A , extends some known cases in the literature. The first example corresponds to the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that we have

$$\begin{cases} -\Delta u = v^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ -\Delta v = u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \\ u, v \in D^{1,2}(\mathbb{R}^N). \end{cases} \quad (1.6)$$

This is a case in which the powers of the non-linearity belong to the so-called *critical hyperbola* introduced by Mitidieri in [M1] and [M2] (see also [CDM] and [PV]). It is an interesting open problem to determine whether the system (1.6) admits nontrivial solutions.

Among the other results, we will show that the trivial solution (U, U) to (1.6) is non-degenerate, up to translations and dilations (see Theorem 1.2 and the ensuing discussion on page 6).

Another interesting system which has a lot of similarities with (1.1) is the well known *Toda system*, namely

$$\begin{cases} -\Delta u_i = \sum_{j=1}^k c_{ij} e^{u_j} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_j} < +\infty, \end{cases} \quad (1.7)$$

for $i = 1, \dots, k$, where $C = (c_{ij})_{i,j=1,\dots,k}$ is the Cartan matrix given by

$$C = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Since we are considering a problem in the plane, the exponential nonlinearity is the natural equivalent. There is a huge literature about the Toda system, we just recall the classification result of [JW] and the paper [GGW] where the matrix $C = \begin{pmatrix} 2 & \mu \\ \mu & 2 \end{pmatrix}$ was considered for $\mu \in (-2, 0)$. Observe that if we consider the Toda system with a general symmetric, invertible, irreducible matrix C and assume that all the entries c_{ij} are positive, then (see [CK, CSW, LZ1, LZ2]) all solutions are radial. In analogy with this result, we believe that if all entries of A are positive then all solutions to (1.1) are radial. We do not investigate this problem in this paper (even if we think that this problem deserves to be studied in the future) and allow some coefficients a_{ij} to be negative. We will see that this causes the solution set to have a richer structure.

1.2. Main results and idea of the proof. A basic remark is that the system (1.1) does not have a *variational structure* and so we cannot apply variational methods. Moreover the critical powers induce a *lack of compactness*. Our basic tool will be the *bifurcation*

theory. Solutions to (1.1) are zeros of the functional $F = (F_1, \dots, F_k) : (D^{1,2}(\mathbb{R}^N))^k \rightarrow (D^{1,2}(\mathbb{R}^N))^k$ defined by

$$F_i(u_1, \dots, u_k) = u_i - (-\Delta)^{-1} \left(\sum_{j=1}^k a_{ij} (u_j^+)^{\frac{N+2}{N-2}} \right) \quad (1.8)$$

for $i = 1, \dots, k$. Of course we have that $F_i(U, \dots, U) = 0$ and our aim is to find solutions close to (U, \dots, U) for suitable values of (a_{ij}) . We will use the classical *Crandall-Rabinowitz theorem* [CR]. Its application requires three basic ingredients:

- (i) a good functional setting for the operator F ,
- (ii) a 1-dimensional kernel for the linearized operator F' ,
- (iii) a transversality condition.

The lack of compactness and the rich structure of the kernel of the linearized operator (see Proposition 1.2 below) make conditions (i) and (ii) not easy to check. (Condition (iii) will be a straightforward computation involving the Jacobi polynomials). Now we discuss the main points and the difficulties to be overcome in (i) and (ii).

Let us start with the functional setting.

First of all let us note that it is not immediate to derive that our solutions are positive. Indeed, since some of the entries a_{ij} are not necessarily positive, we cannot apply the maximum principle. And even if they were, F defined in (1.8) is not smooth enough because $D^{1,2} \rightarrow D^{1,2} : u \mapsto u^+$ is not differentiable. This problem will be solved by restricting our operator to the subspace of $D^{1,2}(\mathbb{R}^N)$ of functions decaying as $|x|^{2-N}$ at infinity. This choice, if from one side will allow to establish the positivity of the solution, on the other hand creates problems to prove the compactness of the linearized operator. This will be discussed in Section 2.2.

Now we discuss the point (ii), i.e., the linearization of F around the trivial solution (U, \dots, U) . This leads to study the problem,

$$\begin{cases} -\Delta v_i = \frac{N(N+2)}{(1+|x|^2)^2} \sum_{j=1}^k a_{ij} v_j & \text{in } \mathbb{R}^N, \\ v_i \in D^{1,2}(\mathbb{R}^N), \end{cases} \quad (1.9)$$

for $i = 1, \dots, k$. It will be shown that (1.9) can be reduced to the classification of eigenvalues and eigenfunctions of the linearized problem associated to the critical equation (1.4) at the standard bubble U , namely,

$$\begin{cases} -\Delta w = \lambda \frac{N(N+2)}{(1+|x|^2)^2} w & \text{in } \mathbb{R}^N, \\ w \in D^{1,2}(\mathbb{R}^N). \end{cases} \quad (1.10)$$

It is known that $\lambda_0 = \frac{N-2}{N+2}$ and $\lambda_1 = 1$ but nothing is known about the other eigenvalues. Our first result completely describes problem (1.10). We believe that this result has its own independent interest.

Theorem 1.1. *The eigenvalues of Problem (1.10) are the numbers*

$$\lambda_n = \frac{(2n + N - 2)(2n + N)}{N(N + 2)}, \quad n \geq 0. \quad (1.11)$$

Each eigenvalue λ_n has multiplicity

$$m(\lambda_n) = \sum_{h=0}^n \frac{(N + 2h - 2)(N + h - 3)!}{(N - 2)! h!} \quad (1.12)$$

and the corresponding eigenfunctions are, in radial coordinates, linear combinations of

$$w_{n,h}(r, \theta) = \frac{r^h}{(1 + r^2)^{h + \frac{N-2}{2}}} P_{n-h}^{(h + \frac{N-2}{2}, h + \frac{N-2}{2})} \left(\frac{1 - r^2}{1 + r^2} \right) Y_h(\theta) \quad (1.13)$$

for $h = 0, \dots, n$, where $Y_h(\theta)$ are spherical harmonics related to the eigenvalue $h(h + N - 2)$ and $P_j^{(\beta, \gamma)}$ are the Jacobi polynomials.

This result extends Theorem 11.1 in [GG] where the linearized problem of the Liouville equation in \mathbb{R}^2 at the standard bubble was considered and highlights the role of the Jacobi polynomials as extension of the Legendre polynomials.

Theorem 1.1 will be used to describe all solutions to (1.9) thanks to a change of variables to diagonalize A . Since A is symmetric, it possesses k real eigenvalues $\Lambda_1, \dots, \Lambda_k$, counting algebraic multiplicity. Assumption (1.3) implies that 1 is always an eigenvalue of A with (at least) the eigenvectors spanned by $(1, \dots, 1)$. So, without loss of generality, we can set $\Lambda_1 = 1$. We have the following result,

Proposition 1.2. *Equation (1.9) possesses a solution $v = (v_1, \dots, v_k) \neq (0, \dots, 0)$ if and only if*

$$\Lambda_i = \lambda_n := \frac{(2n + N - 2)(2n + N)}{N(N + 2)} \quad (1.14)$$

for some $i \in \{1, \dots, k\}$ and $n \in \mathbb{N}$. The solutions coming from (1.14) are given by

$$v = \sum_{h=0}^n c_h w_{n,h}, \quad (1.15)$$

where $c_h \in \mathbb{R}^k$ satisfy $Ac_h = \lambda_n c_h$ and $w_{n,h}$ are defined by (1.13). If several equalities of the form (1.14) hold at the same time, the set of solutions to (1.9) is the linear span of the associated solutions of the form (1.15).

In particular, one always has $\Lambda_1 = 1 = \lambda_1$ and the corresponding solutions are given by

$$v = c_0 \left(x \cdot \nabla U + \frac{N-2}{2} U \right) + \sum_{i=1}^N c_i \frac{\partial U}{\partial x_i} \quad (1.16)$$

for some $c_0, c_1, \dots, c_N \in \mathbb{R}^k$ such that $Ac_i = c_i$ for all $i \in \{0, 1, \dots, N\}$.

To apply Crandall-Rabinowitz theorem, let us consider a \mathcal{C}^1 -path of matrices

$$I \subseteq \mathbb{R} \rightarrow \mathbb{R}^{k \times k} : \alpha \mapsto A(\alpha)$$

such that, for all $\alpha \in I$, $A(\alpha)$ satisfies (1.2)–(1.3). How to deal with more general situations involving two or more parameters, can be found for example in the book [K].

Let $\Lambda_1(\alpha) = 1, \Lambda_2(\alpha), \dots, \Lambda_k(\alpha)$ be the eigenvalues of $A(\alpha)$. The previous result shows that the linearized system (1.9) has two types of degeneracies. The first one, which holds for every value of α , is due to the invariance of the problem (1.1) under dilations and translations, while the second one appears only at the special values α that satisfy (1.14).

Note that, if $n = 0$ (resp. $n = 1$) in (1.14), i.e., if $\Lambda_i(\alpha) = \frac{N-2}{N+2}$ (resp. $\Lambda_i(\alpha) = 1$), then $v = cU$ (resp. $v = c_0(x \cdot \nabla U + \frac{N-2}{2}U) + \sum c_j \frac{\partial U}{\partial x_j}$) are solutions. These are the trivial values of α and they do not provide new solutions to problem (1.1).

Thus (1.14) with $n \geq 2$ is a necessary condition to guarantee bifurcating branches of solutions. A similar phenomenon was previously observed in [GGW] for a general 2×2 Toda system in \mathbb{R}^2 . Coming back to Problem (1.6), we have $\Lambda_1 = 1$ and $\Lambda_2 = -1$, so (1.14) is never satisfied if $n \neq 1$. Hence the solution (U, U) is non-degenerate (up to dilation and translation).

Proposition 1.2 says that the kernel of the linearized operator is composed both by radial and non-radial eigenfunctions. Of course this is a great obstruction to applying Crandall-Rabinowitz Theorem for which a one dimensional kernel is required. For this reason we restrict the problem to the case of radial solutions in $D^{1,2}(\mathbb{R}^N)$, i.e., we work in the space $D_{\text{rad}}^{1,2}(\mathbb{R}^N)$. However we think that the existence of non-radial eigenfunctions implies a *non-radial* bifurcation from the trivial solution. This open problem will be investigated in the future.

In the radial setting (see Corollary 2.2), the kernel of the linearized operator (1.9) has a lower dimension. Its dimension however depends on A . Here we summarize some sufficient conditions on the matrix A such that, in the radial setting, the kernel of the linearized operator is two-dimensional.

Assumptions on the matrix A

Let us suppose that A satisfies (1.2) and (1.3). Moreover assume that there exist $\bar{\alpha}$ and $\bar{i} \in \{2, \dots, k\}$ such that

$$\Lambda_{\bar{i}}(\bar{\alpha}) = \lambda_n \text{ for some } n \geq 2 \text{ (see (1.14) for the definition of } \lambda_n), \quad (1.17)$$

$$\Lambda_j(\bar{\alpha}) \neq \lambda_n \text{ for any } j \notin \{1, \bar{i}\} \text{ and any } n \in \mathbb{N}, \quad (1.18)$$

$$\frac{d\Lambda_{\bar{i}}}{d\alpha}(\bar{\alpha}) \neq 0. \quad (1.19)$$

Some comments on the previous assumptions: (1.18) implies that $\Lambda_{\bar{i}}(\bar{\alpha})$ is simple and then the function $\alpha \mapsto \Lambda_{\bar{i}}(\alpha)$ is smooth in a neighborhood of $\bar{\alpha}$ (see [S] for example). Assumption (1.19) can be reformulated in terms of $\frac{dA}{d\alpha}$. Indeed, if $e_n \neq 0$ is a unit eigenvector of $A(\bar{\alpha})$ for the eigenvalue λ_n , it is not difficult to show that

$$e_n \cdot \frac{dA}{d\alpha}(\bar{\alpha})e_n = \frac{d\Lambda_{\bar{i}}}{d\alpha}(\bar{\alpha}). \quad (1.20)$$

Assumption (1.18) also implies that the kernel of the linearized operator, in this radial setting, is two-dimensional and (1.19) gives the *transversality* condition of Crandall and Rabinowitz, see [CR]. Finally, constraining further our operator to the orthogonal space with the radial function $W(|x|) = \frac{1-|x|^2}{(1+|x|^2)^{N/2}}$, we get a one dimensional kernel and so Crandall-Rabinowitz Theorem applies. Then the point $(\bar{\alpha}, U, \dots, U)$ is a bifurcation

point when $\bar{\alpha}$ satisfies $\Lambda_{\bar{i}}(\bar{\alpha}) = \lambda_n$ for some $\bar{i} = 2, \dots, k$ and some $n \geq 2$. However this construction produces a *Lagrange multiplier* for the equations satisfied by the zeros of F .

The last step is to show that this Lagrange multiplier is 0. In our opinion this is one of the interesting points of the paper and it will be done using a suitable version of the Pohozaev identity.

The Pohozaev identity was used by many authors dealing with systems of just two equations. The extension to the case of more equations is not straightforward and requires the additional assumption of the invertibility of the matrix A (see section 3.3). Now we are in position to state our bifurcation result.

Theorem 1.3. *If A satisfies (1.2) and (1.3) and if $\bar{\alpha}$ and \bar{i} verify the assumptions (1.17)–(1.19) and if*

$$\text{the matrix } A(\bar{\alpha}) \text{ is invertible,} \quad (1.21)$$

then the point $(\bar{\alpha}, U, \dots, U)$ is a radial bifurcation point for the curve of trivial solutions $\alpha \mapsto (\alpha, U, \dots, U)$ to equation (1.1). More precisely, there exist a continuously differentiable curve defined for ε small enough

$$(-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R} \times (D_{\text{rad}}^{1,2}(\mathbb{R}^N))^k : \varepsilon \mapsto (\alpha(\varepsilon), u_1(\varepsilon), \dots, u_k(\varepsilon)) \quad (1.22)$$

emanating from $(\bar{\alpha}, U, \dots, U)$, i.e., $(\alpha(0), u_1(0), \dots, u_k(0)) = (\bar{\alpha}, U, \dots, U)$, such that, for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $u(\varepsilon) = (u_1(\varepsilon), \dots, u_k(\varepsilon))$ is a radial solution to

$$\begin{cases} -\Delta u_i = \sum_{j=1}^k a_{ij}(\alpha) u_j^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ u_i > 0 & \text{in } \mathbb{R}^N, \\ u_i \in D^{1,2}(\mathbb{R}^N), \end{cases} \quad (1.23)$$

with $\alpha = \alpha(\varepsilon)$. Moreover

$$u(\varepsilon) = (1, \dots, 1)U + \varepsilon e_n W_n(|x|) + \varepsilon \varphi_\varepsilon(|x|) \quad (1.24)$$

where e_n is an eigenvector of A for the eigenvalue $\Lambda_{\bar{i}}(\bar{\alpha}) = \lambda_n$, W_n is the function defined in (2.12), and φ_ε is an uniformly bounded function in $(D^{1,2}(\mathbb{R}^N))^k$ such that $\varphi_0 = 0$.

Now let us discuss the case $k = 2$. Here the number of degree of freedom is $(k-1)k/2 = 1$ and the matrix A depends on a single parameter: $A = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}$. Its eigenvalues are given by $\Lambda_1 = 1$, $\Lambda_2 = 2\alpha - 1$. It is easily seen that (1.17)–(1.19) are verified with $\bar{i} = 2$ and, in view of (1.14), with $\alpha = \bar{\alpha}_n$ satisfying

$$\bar{\alpha}_n = \frac{2n^2 + 2Nn - 2n + N^2}{N(N+2)} \quad (1.25)$$

so that the degeneracy occurs at a sequence of values α_n such that $\alpha_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Note that $\bar{\alpha}_n \neq \frac{1}{2}$ for any $n \in \mathbb{N}$, which implies (1.21). Hence Theorem 1.3 holds at the values α_n without additional assumptions and it becomes

Theorem 1.4. *If $A = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}$, then, for any $n \geq 2$, the points $(\bar{\alpha}_n, U, U)$ are radial bifurcation points for the curve of trivial solutions (α, U, U) to equation (1.1). More precisely, there exist a continuously differentiable curve defined for ε small enough*

$$(-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R} \times (D_{\text{rad}}^{1,2}(\mathbb{R}^N))^2 : \varepsilon \mapsto (\alpha(\varepsilon), u_1(\varepsilon), u_2(\varepsilon)) \quad (1.26)$$

passing through $(\bar{\alpha}_n, U, U)$, i.e., $(\alpha(0), u_1(0), u_2(0)) = (\bar{\alpha}_n, U, U)$, such that, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $u_i(\varepsilon)$ is a radial solution to

$$\begin{cases} -\Delta u_1 = \alpha u_1^{\frac{N+2}{N-2}} + (1-\alpha)u_2^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = (1-\alpha)u_1^{\frac{N+2}{N-2}} + \alpha u_2^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ u_1, u_2 > 0 & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases} \quad (1.27)$$

with $\alpha = \alpha(\varepsilon)$. Moreover,

$$\begin{aligned} u_1(\varepsilon) &= U + \varepsilon W_n(|x|) + \varepsilon \varphi_{1,\varepsilon}(|x|) \\ u_2(\varepsilon) &= U - \varepsilon W_n(|x|) + \varepsilon \varphi_{2,\varepsilon}(|x|) \end{aligned} \quad (1.28)$$

where W_n is the function defined in (2.12), and $\varphi_{1,\varepsilon}, \varphi_{2,\varepsilon}$ are functions uniformly bounded in $D^{1,2}(\mathbb{R}^N)$ and such that $\varphi_{s,0} = 0$ for $s = 1, 2$.

The same type of result was proved in [GGW] for the Toda system in \mathbb{R}^2 .

1.3. Extensions and related problems. A first interesting question that arises from Proposition 1.2 concerns the existence of non-radial solutions.

Question 1. *Do exist nonradial solutions to (1.1) bifurcating from the values α which verify (1.14)? How many?*

In analogy with the classification result of Jost-Wang ([JW]), it is possible to think that the number of solutions of (1.1) coincides with that of the linearized operator (see also [WZZ]). In our case, if for example $k = 2$, we would have the existence of at least $\sum_{h=0}^n \frac{(N+2h-2)(N+h-3)!}{(N-2)!h!}$ distinct solutions.

Another interesting question concerns the shape of the branch of our solutions.

Question 2. *What about the bifurcation diagram for $\alpha(\varepsilon)$ close to α ?*

This question is quite delicate and the answer seems to strongly depend on A . In Appendix A, we carry out the computation of the first derivative of $\alpha(\varepsilon)$ with respect to ε . It is worth noting that if $k = 2$ then $\frac{d\alpha}{d\varepsilon}(0) = 0$ (then nothing can be said about the behaviour of the branch) but, if $k > 2$, we can have that $\frac{d\alpha}{d\varepsilon}(0) \neq 0$. In this case, the bifurcation is *transcritical*.

Question 3. What possible extensions may be considered?

Another interesting problem to which one can apply our techniques is given by the Gross-Pitaevskii type systems, namely,

$$\begin{cases} -\Delta u_i = \left(\sum_{j=1}^k a_{ij} u_j^2 \right)^{\frac{2}{N-2}} u_i & \text{in } \mathbb{R}^N, \\ u_i > 0 & \text{in } \mathbb{R}^N, \\ u_i \in D^{1,2}(\mathbb{R}^N), \end{cases} \quad (1.29)$$

When $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, this problem was studied in [DH] as the limit problem for blowing up solution on Riemannian manifolds. The authors proved that only the trivial solution (U, \dots, U) exist.

The paper is organized as follows. In Section 2, we prove Theorem 1.1 and we study the linearization of our system at the trivial solution. In Section 3 we define the functional setting, we apply the Crandall-Rabinowitz result, and we prove the Pohozaev identity getting our bifurcation result, Theorem 1.3. Finally in the appendix, we give some examples of possible behaviour of the branches.

2. LINEARIZATION AT THE STANDARD BUBBLE

2.1. The case of a single equation. Let us consider the critical equation (1.4) and the associated eigenvalue problem (1.10). It is well known that the first eigenvalue to (1.10) is given by $\lambda_0 = \frac{N-2}{N+2} < 1$ and the corresponding eigenfunction is U , while the second eigenvalue is $\lambda_1 = 1$ and it has an $N + 1$ dimensional kernel spanned by $\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_N}, x \cdot \nabla U + \frac{N-2}{2}U$ (see for example [BE] or [AGAP]). In this section, we compute all eigenvalues and corresponding eigenfunctions of (1.10).

Proof of Theorem 1.1. We decompose the solutions to (1.10) using spherical harmonics:

$$w(r, \theta) = \sum_{h=0}^{\infty} \psi_h(r) Y_h(\theta), \quad \text{where } r = |x|, \theta = \frac{x}{|x|} \in \mathbb{S}^{N-1},$$

and

$$\psi_h(r) = \int_{\mathbb{S}^{N-1}} w(r, \theta) Y_h(\theta) d\theta.$$

Here $Y_h(\theta)$ denotes a h -th spherical harmonic which satisfies:

$$-\Delta_{\mathbb{S}^{N-1}} Y_h = \beta_h Y_h$$

where $\Delta_{\mathbb{S}^{N-1}}$ is the Laplace-Beltrami operator on \mathbb{S}^{N-1} with the standard metric and β_h is the h -th eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}$. It is known that

$$\beta_h = h(N - 2 + h), \quad h = 0, 1, 2, \dots$$

and its multiplicity is

$$m(\beta_h) = \frac{(N + 2h - 2)(N + h - 3)!}{(N - 2)! h!}.$$

By standard regularity theory, the function $w \in D^{1,2}(\mathbb{R}^N)$ is a solution of (1.10) if and only if $\psi_h(r)$ is a weak solution of

$$\begin{cases} -\psi_h''(r) - \frac{N-1}{r}\psi_h'(r) + \frac{\beta_h}{r^2}\psi_h(r) = \lambda \frac{N(N+2)}{(1+r^2)^2} \psi_h(r), & \text{in } (0, \infty) \\ \int_0^{+\infty} r^{N-1} |\psi_h'(r)|^2 dr < +\infty \end{cases} \quad (2.1)$$

We shall solve (2.1) using the following transformation

$$\psi_h(r) = r^{-\frac{N-2}{2}} B_h(r).$$

The function B_h solves

$$-B_h''(r) - \frac{1}{r}B_h'(r) = -\nu_h^2 \frac{1}{r^2} B_h(r) + \frac{\lambda N(N+2)}{(1+r^2)^2} B_h(r) \quad \text{in } (0, \infty) \quad (2.2)$$

where $\nu_h := h + \frac{N-2}{2}$, and

$$\int_0^\infty r (B_h'(r))^2 + r^{-1} (B_h(r))^2 dr < \infty. \quad (2.3)$$

The proof of (2.2) is a straightforward computation, so let us prove (2.3). By the definition of ψ_h we get

$$r^{N-1} |\psi_h'|^2 = r (B_h')^2 + \left(\frac{N-2}{2} \right)^2 \frac{B_h^2}{r} - (N-2) B_h' B_h.$$

Now, integrating between ε_n and R_n (the sequences ε_n and R_n will be chosen later) we get

$$\int_{\varepsilon_n}^{R_n} r^{N-1} |\psi_h'|^2 dr = \int_{\varepsilon_n}^{R_n} r (B_h')^2 + \left(\frac{N-2}{2} \right)^2 \frac{B_h^2}{r} dr - \frac{N-2}{2} (B_h^2(R_n) - B_h^2(\varepsilon_n)).$$

Since $\psi_h \in L^{2^*}([0, \infty); r^{N-1} dr)$, there exist sequences $\varepsilon_n \rightarrow 0$ and $R_n \rightarrow +\infty$ such that

$$R_n^{\frac{N-2}{2}} |\psi_h(R_n)| \rightarrow 0 \text{ and } \varepsilon_n^{\frac{N-2}{2}} |\psi_h(\varepsilon_n)| \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (2.4)$$

Finally, by (2.4), we deduce that $B_h(R_n)$ and $B_h(\varepsilon_n)$ go to zero and this, together with (2.1), gives (2.3).

Then, from Lemma 2.4 of [GGN], we have that $B_h(0) = 0$ and

$$B_h(r) = O(r^{\nu_h}) \quad \text{as } r \rightarrow 0. \quad (2.5)$$

Let us show that an analogous estimate holds at infinity. To do this set $C(r) := B_h(\frac{1}{r})$. So the claim follows if we prove that C is bounded near the origin. A straightforward computation proves that C satisfies again (2.2) and (2.3). Then by (2.28) of Lemma 2.4 of [GGN] we get that

$$C(r) = O(r^{\nu_h}) \quad \text{as } r \rightarrow 0,$$

which gives

$$B_h(r) = O(r^{-\nu_h}) \quad \text{as } r \rightarrow +\infty \quad (2.6)$$

and so the claim follows.

Setting $\xi := \frac{1-r^2}{1+r^2}$, letting $R_h(\xi) := B_h(r)$, and using the definition of β_h we have

$$\frac{\partial}{\partial \xi} \left((1 - \xi^2) \frac{\partial R_h}{\partial \xi} \right) + \left(\lambda \frac{N(N+2)}{4} - \frac{\nu_h^2}{1 - \xi^2} \right) R_h(\xi) = 0 \quad (2.7)$$

for $-1 < \xi < 1$. Now, let us set $A_h(\xi) := (1 - \xi^2)^{-\nu_h/2} R_h(\xi)$. Then $A_h(\xi)$ solves, for $\xi \in (-1, 1)$,

$$(1 - \xi^2) A_h''(\xi) - 2(1 + \nu_h) \xi A_h'(\xi) + \left[\frac{N(N+2)}{4} \lambda - \nu_h^2 - \nu_h \right] A_h(\xi) = 0. \quad (2.8)$$

This is a particular case of the Jacobi equation

$$(1 - \xi^2) y'' + [\gamma - \beta - (2 + \beta + \gamma) \xi] y' + m(1 + \beta + \gamma + m) y = 0$$

with $\beta = \gamma = \nu_h$. From (2.5) we have that $r^{-\nu_h} B_h(r) = R_h(\xi) \left(\frac{1+\xi}{1-\xi} \right)^{\nu_h/2}$ is bounded near $\xi = 1$, while from (2.6) we have that $r^{\nu_h} B_h(r) = R_h(\xi) \left(\frac{1-\xi}{1+\xi} \right)^{\nu_h/2}$ is bounded near $\xi = -1$. This implies that the solution $A_h(\xi)$ of (2.8) is bounded at $\xi = \pm 1$.

It is known that (2.8) admits a bounded solution if and only if

$$\frac{N(N+2)}{4} \lambda - \nu_h^2 - \nu_h = m(m + 2\nu_h + 1)$$

for $m = 0, 1, \dots$. Inserting the definition of ν_h we get that

$$\lambda = \frac{4}{N(N+2)} \left((m+h)^2 + (m+h)(N-1) + \frac{N(N-2)}{4} \right)$$

and, setting $n := m + h \in \mathbb{N}$, we derive

$$\lambda = \frac{(2n + N - 2)(2n + N)}{N(N + 2)}.$$

Moreover the bounded solutions of (2.8) corresponding to $m(m + 2\nu_h + 1)$ are given by multiples of $A_{m,h}(\xi) = P_m^{(\nu_h, \nu_h)}(\xi)$ where $P_m^{(\gamma, \beta)}$ are the Jacobi polynomials:

$$P_m^{(\beta, \gamma)}(\xi) = \sum_{s=0}^m \binom{m+\beta}{s} \binom{m+\gamma}{m-s} \left(\frac{\xi-1}{2} \right)^{m-s} \left(\frac{\xi+1}{2} \right)^s.$$

The polynomials $P_m^{(\nu_h, \nu_h)}$ are also known as the *Gegenbauer polynomials* or the *ultra-spherical polynomials*. They form a basis of the space $L^2((-1, 1); (1 - \xi^2)^{\nu_h} d\xi)$ (see p. 202 in [LTWZ] for example). Moreover we have that

$$\begin{aligned} R_{m,h}(\xi) &= (1 - \xi^2)^{\nu_h/2} P_m^{(\nu_h, \nu_h)}(\xi), \\ B_{m,h}(r) &= \frac{(2r)^{\nu_h}}{(1 + r^2)^{\nu_h}} P_m^{(\nu_h, \nu_h)} \left(\frac{1 - r^2}{1 + r^2} \right) \end{aligned}$$

and

$$\psi_{m,h}(r) = \frac{2^{h+\frac{N-2}{2}} r^h}{(1 + r^2)^{h+\frac{N-2}{2}}} P_m^{(h+\frac{N-2}{2}, h+\frac{N-2}{2})} \left(\frac{1 - r^2}{1 + r^2} \right).$$

Finally, recalling that $n = m + h$ then (1.13) follows. The multiplicity (1.12) then follows counting the multiplicity of the spherical harmonics. \square

Remark 2.1. Note that Theorem 1.1 also holds when $N = 2$. In this case we have that $\lambda_n = \frac{n(n+1)}{2}$. These are exactly the eigenvalues associated to the linearization of the classical Liouville problem

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^U dx < \infty \end{cases}$$

at the standard bubble $U_{\mathbb{R}^2}(x) = \log \frac{64}{(8+|x|^2)^2}$. This result was proved in [GG] (see Theorem 11.1) and the corresponding eigenfunctions are spanned by $(P_L)_{n-h}^h \left(\frac{8-r^2}{8+r^2} \right) Y_h(\theta)$ where (P_L) are the Legendre polynomials and $Y_h(\theta)$ are the spherical harmonics in \mathbb{R}^2 . Observe that for $N = 2$ the Jacobi polynomials $P_{n-h}^{(h+\frac{N-2}{2}, h+\frac{N-2}{2})} \left(\frac{1-r^2}{1+r^2} \right)$ in (1.13) become the Legendre polynomials and so Theorem 1.1 contains also the result of [GG] for \mathbb{R}^2 .

2.2. The case of the system. Now we are in position to prove Proposition 1.2 in the Introduction.

Proof of Proposition 1.2. Because A is symmetric, there exists an orthogonal matrix $B = (b_{ij})$ such that

$$B^{-1}AB = \Lambda \quad (2.9)$$

and Λ is the diagonal matrix with the eigenvalues $(\Lambda_1, \dots, \Lambda_k)$ as diagonal elements. Let $w_i := \sum_{j=1}^k b_{ij}^{-1} v_j = \sum_{j=1}^k b_{ji} v_j$. Then $w = (w_1, \dots, w_k)$ is a solution to

$$\forall i = 1, \dots, k, \quad \begin{cases} -\Delta w_i = \frac{N(N+2)}{(1+|x|^2)^2} \Lambda_i w_i & \text{in } \mathbb{R}^N, \\ w_i \in D^{1,2}(\mathbb{R}^N). \end{cases} \quad (2.10)$$

By our choice of the matrix B in (2.9) the k equations in (2.10) are decoupled and so we can solve them independently. Remember that we have set $\Lambda_1 = 1$. So the first equation of (2.10) reads

$$-\Delta w_1 = \frac{N(N+2)}{(1+|x|^2)^2} w_1 \quad \text{in } \mathbb{R}^N \quad (2.11)$$

and it is well known it admits a nontrivial solution which is a linear combination of $\frac{\partial U}{\partial x_i}$ for $i = 1, \dots, N$ and $x \cdot \nabla U + \frac{N-2}{2} U$. It is important to observe that equation (2.11) does not depend on α and so system (2.10) has the solution in (1.16) for every value of α .

The other equations in (2.10) have a nontrivial solution if and only if Λ_i is an eigenvalue of problem (1.10), i.e., using Theorem 1.1 if and only if (1.14) is satisfied for some $i = 2, \dots, k$, for some $n \in \mathbb{N}$ and for some value of $\alpha \in \mathbb{R}$. When (1.14) is satisfied the i -th equation in system (2.10) has as a solutions a linear combination of the eigenfunctions of (1.10) related to the eigenvalue λ_n , and so (1.15) follows. \square

One of the main hypothesis of the bifurcation result is that the kernel of the linearized operator has to be one dimensional. From the previous result, we know that

the linearized operator has instead a very rich kernel. To overcome this problem we consider only the case of radial solutions in $D^{1,2}(\mathbb{R}^N)$, that is, we will work in the space $D_{\text{rad}}^{1,2}(\mathbb{R}^N)$. First we state the result of Proposition 1.2 in this radial setting.

Corollary 2.2. *Equation (1.9) possesses a radial solution $v = (v_1, \dots, v_k) \neq (0, \dots, 0)$ if and only if $\Lambda_i = \lambda_n$ for some $i \in \{1, \dots, k\}$ and $n \in \mathbb{N}$, in which case the associated radial solutions are given by $v = cW_n$ where $c \in \mathbb{R}^k$ is an eigenvector associated to the eigenvalue Λ_i and*

$$W_n(|x|) := \frac{1}{(1 + |x|^2)^{\frac{N-2}{2}}} P_n^{\left(\frac{N-2}{2}, \frac{N-2}{2}\right)} \left(\frac{1 - |x|^2}{1 + |x|^2} \right). \quad (2.12)$$

Here $P_n^{\left(\frac{N-2}{2}, \frac{N-2}{2}\right)}(\xi) = \sum_{s=0}^n \binom{n + \frac{N-2}{2}}{s} \binom{n + \frac{N-2}{2}}{n-s} \left(\frac{\xi-1}{2}\right)^{n-s} \left(\frac{\xi+1}{2}\right)^s$. If several equalities of the form $\Lambda_i = \lambda_n$ hold at the same time, the set of radial solutions to (1.9) is the linear span of the associated solutions of the form (2.12).

In particular, if $\Lambda_1 = 1$ is a simple eigenvalue of A (which is the case under assumption (1.18)), all corresponding radial solutions to equation (1.9) are given by multiples of

$$(1, \dots, 1)W \quad \text{where } W(|x|) := x \cdot \nabla U + \frac{N-2}{2}U = d \frac{1 - |x|^2}{(1 + |x|^2)^{N/2}} \quad (2.13)$$

and $d = \frac{1}{2}N^{(N-2)/4}(N-2)^{(N+2)/4}$.

Proof. Restricting to the radial setting we have that equation (2.11) admits only the solution in (2.13). From Theorem 1.1 instead follows that any eigenvalue λ_n of the critical problem admits one radial solution which is the one in (1.13) that correspond to $h = 0$. Then (2.12) follows from (1.15). \square

3. THE BIFURCATION RESULT

3.1. The functional setting. As mentioned in the introduction, the proof of our bifurcation result requires an appropriate functional setting which is a delicate part of the proof.

Both for the lack of differentiability of $u \mapsto u^+$ and the difficulty of proving that the solution (u_1, \dots, u_k) is positive, we need to restrict to a subset of $D^{1,2}(\mathbb{R}^N)$ with a stronger topology. As before let $D_{\text{rad}}^{1,2}(\mathbb{R}^N) = \{u \in D^{1,2}(\mathbb{R}^N) \mid u = u(|x|)\}$ and set

$$D := \left\{ u \in L^\infty(\mathbb{R}^N) \mid \sup_{x \in \mathbb{R}^N} \frac{|u(x)|}{U(x)} < +\infty \right\}$$

endowed with the norm $\|u\|_D := \sup_{x \in \mathbb{R}^N} \frac{|u(x)|}{U(x)}$ and we define

$$X = D_{\text{rad}}^{1,2}(\mathbb{R}^N) \cap D.$$

Then X is a Banach space equipped with the norm $\|u\|_X := \max\{\|u\|_{1,2}, \|u\|_D\}$ where $\|u\|_{1,2}$ is the classical norm on $D^{1,2}(\mathbb{R}^N)$.

To rule out the degeneracy due to the invariance under dilations of Problem (1.1), we will solve the linearized equation in the subspace of functions that are orthogonal in $(D_{\text{rad}}^{1,2}(\mathbb{R}^N))^k$ to $(1, \dots, 1)W(|x|)$ defined in (2.13). Let P_K be the orthogonal projection

(with respect to the inner product of $(D^{1,2}(\mathbb{R}^N))^k$) from X^k onto the subspace K given by

$$K := \left\{ g \in X^k \mid \sum_{i=1}^k \int_{\mathbb{R}^N} \nabla W \cdot \nabla g_i(x) \, dx = 0 \right\}. \quad (3.1)$$

Definition 3.1. Let us denote by $B := \{u \in X \mid \|u - U\|_X < \frac{1}{2}\}$ and define the operator

$$T : \mathbb{R} \times (K \cap B^k) \rightarrow K$$

as

$$T(\alpha, u_1, \dots, u_k) := P_K \begin{pmatrix} u_1 - (-\Delta)^{-1} \sum_{j=1}^k a_{1j}(\alpha) u_j^{2^*-1} \\ \vdots \\ u_k - (-\Delta)^{-1} \sum_{j=1}^k a_{kj}(\alpha) u_j^{2^*-1} \end{pmatrix} \quad (3.2)$$

Note that, since $u_i \in B$, $u_i = U + (u_i - U) > \frac{1}{2}U$ is positive so that $u_i^{2^*-1}$ is well defined for any $N \geq 3$.

The zeros of the operator T satisfy

$$\begin{cases} -\Delta u_i = \sum_{j=1}^k a_{ij}(\alpha) u_j^{2^*-1} + L \frac{N(N+2)}{(1+|x|^2)^2} W & \text{in } \mathbb{R}^N, \text{ for } i = 1, \dots, k, \\ u = (u_1, u_2, \dots, u_k) \in K \cap B^k, \end{cases} \quad (3.3)$$

where $L = L(u) \in \mathbb{R}$ is a Lagrange multiplier. Once we prove the existence of (u_1, \dots, u_k) that satisfies (3.3), the final step will be to show that $L = 0$ so that u is indeed a solution to (1.1) and this will be done in the next section using a *Pohozaev* identity.

First we prove some properties of the operator T .

Lemma 3.2. *The projector $P_K : X \rightarrow X$ is well defined and continuous.*

Proof. Let $W^* := W/\|W\|_{1,2}$. One can write $P_K u = u - (1, \dots, 1)W^* \sum_{i=1}^k (u_i | W^*)_{D^{1,2}}$. From (2.13), one easily shows that there exists a $C \in \mathbb{R}$ such that $|W^*| \leq CU$ and so $\|W^*\|_X < +\infty$. The statement readily follows from these facts. \square

Lemma 3.3. *The operator T in (3.2) is continuous from $\mathbb{R} \times (K \cap B^k)$ into K and its derivatives $\partial_\alpha T$, $\partial_u T$ and $\partial_{\alpha u} T$ exist and are continuous.*

Proof. Since u_j belongs to X , we have that $u_j^{2^*-1} \in L^{\frac{2N}{N+2}}_{\text{rad}}(\mathbb{R}^N)$ for any $j = 1, \dots, k$. As a consequence, there exists a unique $g_i \in D^{1,2}_{\text{rad}}(\mathbb{R}^N)$ for $i = 1, \dots, k$ such that g_i is a weak solution to $-\Delta g_i = f_i$ in \mathbb{R}^N where

$$f_i := \sum_{j=1}^k a_{ij}(\alpha) u_j^{2^*-1} \quad (3.4)$$

The solution g_i enjoys the following representation:

$$g_i(x) = \frac{1}{\omega_N(N-2)} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} f_i(y) dy,$$

where ω_N is the area of the unit sphere in \mathbb{R}^N . By assumption $u_i \in B$ and this implies that $|f_i(x)| \leq CU^{2^*-1}(x)$ so that

$$|g_i(x)| \leq C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} U^{2^*-1}(y) dy = CU(x)$$

and $g_i \in X$. (Different occurrences of C may denote different constants.) To prove the continuity of T in $K \cap B^k$, let $\alpha_n \rightarrow \alpha$ in \mathbb{R} and $u_{i,n} \rightarrow u_i$ in X (for $i = 1, \dots, k$) as $n \rightarrow \infty$, and set

$$g_{i,n} := (-\Delta)^{-1} f_{i,n} \quad \text{where } f_{n,i} := \sum_{j=1}^k a_{ij}(\alpha_n) u_{j,n}^{2^*-1}.$$

Since $u_{i,n} \rightarrow u_i$ in $D^{1,2}(\mathbb{R}^N)$, the convergence also holds in $L^{2^*}(\mathbb{R}^N)$. Using Lebesgue's dominated convergence theorem and its converse, one deduces that $f_{i,n} \rightarrow f_i$ in $L^{\frac{2N}{N+2}}$ where f_i is defined as in (3.4). Therefore $g_{i,n} \rightarrow g_i$ in $D^{1,2}$ and $T(\alpha_n, u_n) \rightarrow T(\alpha, u)$ in $D^{1,2}$.

Now let us show the convergence in D . We have that

$$\begin{aligned} \frac{|g_{i,n}(x) - g_i(x)|}{U(x)} &\leq \frac{1}{\omega_N(N-2)U(x)} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{|f_{i,n}(y) - f_i(y)|}{U(y)^{2^*-1}} U(y)^{2^*-1} dy \\ &\leq C \sup_{y \in \mathbb{R}^N} \frac{|f_{i,n}(y) - f_i(y)|}{U(y)^{2^*-1}}. \end{aligned}$$

Moreover,

$$\begin{aligned} \sup_{y \in \mathbb{R}^N} \frac{|f_{i,n}(y) - f_i(y)|}{U(y)^{2^*-1}} &\leq \sum_{j=1}^k |a_{ij}(\alpha_n) - a_{ij}(\alpha)| \sup_{y \in \mathbb{R}^N} \left(\frac{|u_j(y)|}{U(y)} \right)^{2^*-1} \\ &\quad + \sum_{j=1}^k |a_{ij}(\alpha_n)| \sup_{y \in \mathbb{R}^N} \left| \left(\frac{u_{j,n}(y)}{U(y)} \right)^{2^*-1} - \left(\frac{u_j(y)}{U(y)} \right)^{2^*-1} \right|. \end{aligned}$$

The first term goes to 0 because $a_{ij}(\alpha_n) \rightarrow a_{ij}(\alpha)$ and $|u_j| \leq CU$. As $(a_{ij}(\alpha_n))_n$ are bounded sequences, it is enough to show that the last factor goes to 0. This is the case because, thanks to the convergence in D , $u_{j,n}/U \rightarrow u_j/U$ uniformly for all j and the map $\zeta \mapsto \zeta^{2^*-1}$ is continuous.

The existence of $\partial_\alpha T$, $\partial_u T$ and $\partial_{\alpha u} T$ (for the topology of X) and their continuity follows in a very similar way and we omit it. \square

Next we show a compactness result for the operator $w \mapsto (-\Delta)^{-1} \left(\frac{w}{(1+|x|^2)^2} \right)$. We need some decay estimates on solutions of a semilinear elliptic equation.

Lemma 3.4. *If $0 < p < N$ and h is a nonnegative, radial function belonging to $L^1(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \frac{h(y)}{|x-y|^p} dy = O\left(\frac{1}{|x|^p}\right) \quad \text{as } |x| \rightarrow +\infty.$$

The general statement of this Lemma which also applies in a nonradial setting can be found in [ST]. Here we report only the radial version.

Now we can prove our compactness result:

Lemma 3.5. *The operator*

$$M(w) := (-\Delta)^{-1} \left(\frac{w}{(1+|x|^2)^2} \right) \quad (3.5)$$

is compact from X to X .

Proof. First of all, let us show that M is well defined. If $w \in X$ then $|w| \leq \|w\|_D U$ so that $\frac{|w|}{(1+|x|^2)^2} \leq C\|w\|_D U^{\frac{N+2}{N-2}} \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ and, using the fact that $(-\Delta)^{-1} : L^{\frac{2N}{N+2}} \rightarrow D^{1,2}$, one gets $M(w) \in D^{1,2}$. Moreover

$$|M(w)| \leq C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{|w(y)|}{(1+|y|^2)^2} dy \leq C\|w\|_D \int_{\mathbb{R}^N} \frac{U^{\frac{N+2}{N-2}}(y)}{|x-y|^{N-2}} = C\|w\|_D U(x), \quad (3.6)$$

and so $M(w) \in D$. This argument incidentally shows that $M : X \rightarrow X$ is continuous.

Now let (w_n) be a bounded sequence in X and let us prove that, up to a subsequence, $g_n := M(w_n)$ converges strongly to some $g \in X$. On one hand, since (w_n) is bounded in $D^{1,2}$, going if necessary to a subsequence, one can assume that (w_n) converges weakly to some w in $D^{1,2}$ and $w_n \rightarrow w$ almost everywhere. On the other hand, $(\|w_n\|_D)$ is also bounded which means that $|w_n| \leq CU$ where C is independent of n and so $\frac{|w_n|}{(1+|x|^2)^2} \leq CU^{\frac{N+2}{N-2}}$. Lebesgue's dominated convergence theorem then implies that $\frac{w_n}{(1+|x|^2)^2}$ converges strongly to $\frac{w}{(1+|x|^2)^2}$ in $L^{\frac{2N}{N+2}}$. From the continuity of $(-\Delta)^{-1} : L^{\frac{2N}{N+2}} \rightarrow D^{1,2}$, one concludes that $g_n \rightarrow g$ in $D^{1,2}$. Moreover, passing to the limit on $|w_n| \leq CU$ yields $w \in D$, and passing to the limit on the inequality (3.6) for $w = w_n$ yields $g \in D$.

It remains to show that $\|g_n - g\|_D \rightarrow 0$. This is somewhat similar to the argument used in Lemma 3.3. First, Hölder inequality allows us to get the estimate:

$$\begin{aligned} |g_n(x) - g(x)| &\leq C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{|w_n(y) - w(y)|}{(1+|y|^2)^2} \\ &= C \int_{\mathbb{R}^N} \frac{U^{\frac{N+2}{N-2}-\varepsilon}(y)}{|x-y|^{N-2}} \frac{|w_n(y) - w(y)|}{U^{1-\varepsilon}(y)} \\ &\leq C \left(\int_{\mathbb{R}^N} \left| \frac{U^{\frac{N+2}{N-2}-\varepsilon}(y)}{|x-y|^{N-2}} \right|^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \cdot \left(\int_{\mathbb{R}^N} \left| \frac{|w_n(y) - w(y)|}{U^{1-\varepsilon}(y)} \right|^q \right)^{\frac{1}{q}} \end{aligned} \quad (3.7)$$

where $\varepsilon > 0$ will be chosen small and q large and satisfying $\varepsilon q = \frac{2N}{N-2}$. Because of this latter constraint, the integrand of the right integral is bounded by $\|w_n - w\|_D^q U^{\varepsilon q}(y) \leq C U^{\varepsilon q}(y) \in L^1$ where C is independent of n . Lebesgue's dominated convergence theorem then implies that this integral converges to 0 as $n \rightarrow \infty$.

The proof will be complete if we show:

$$\int_{\mathbb{R}^N} \left| \frac{U^{\frac{N+2}{N-2}-\varepsilon}(y)}{|x-y|^{N-2}} \right|^{\frac{q}{q-1}} dy \leq \frac{C}{(1+|x|)^{(N-2)\frac{q}{q-1}}} = C U^{\frac{q}{q-1}}. \quad (3.8)$$

This inequality follows from Lemma 3.4 because $h(y) = U^{\left(\frac{N+2}{N-2}-\varepsilon\right)\frac{q}{q-1}} \in L^1$ if and only if $\left(\frac{N+2}{N-2}-\varepsilon\right)\frac{q}{q-1} > \frac{N}{N-2}$, which is possible if ε is small enough and q is large enough. \square

3.2. Application of the Crandall-Rabinowitz Theorem. In this section we will verify the assumptions of the of the Crandall-Rabinowitz Theorem. Let us recall that by Corollary 2.2, the linearized system (1.9) has the following radial solutions

- i) $(1, \dots, 1)W$ (due to the dilation invariance of the problem), for every α ,
- ii) $\eta := e_n W_n(|x|)$ where $e_n \neq 0$ satisfies $A(\bar{\alpha})e_n = \lambda_n e_n$ (see (2.12)) for $\bar{\alpha}$ satisfying (1.14).

Notice that $(1, \dots, 1) \perp e_n$ and so $(1, \dots, 1)W \perp \eta$ in $(D^{1,2})^k$, i.e., $\eta \in K$.

To apply Rabinowitz' result, we need to verify the assumptions of Theorem 1.7 in [CR]. This is the purpose of the following lemmas.

Lemma 3.6. *Let T be as defined in (3.2) and assume that $\bar{\alpha}$ satisfies (1.17)–(1.18). Then $\ker(\partial_u T(\bar{\alpha}, U, \dots, U))$ is one dimensional and it is given by*

$$\ker(\partial_u T(\bar{\alpha}, U, \dots, U)) = \text{span}\{\eta\} \quad \text{where } \eta = e_n W_n, \quad (3.9)$$

W_n is defined in (2.12), $e_n \neq 0$, and $A(\bar{\alpha})e_n = \lambda_n e_n$.

Proof. Let us consider the Fréchet derivative of T at (α, U, \dots, U) . We have that

$$\partial_u T(\alpha, U, \dots, U) \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = P_K \left(w_i - (-\Delta)^{-1} \left(\sum_{j=1}^k a_{ij}(\alpha) \frac{N(N+2)}{(1+|x|^2)^2} w_j \right) \right)_{i=1}^k \quad (3.10)$$

so that $\partial_u T(\bar{\alpha}, U, \dots, U) \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ if and only if $(w_1, \dots, w_k) \in K$ is a solution to

$$\forall i = 1, \dots, k, \quad -\Delta w_i - \frac{N(N+2)}{(1+|x|^2)^2} \sum_{j=1}^k a_{ij}(\bar{\alpha}) w_j = -L \Delta W \quad \text{in } \mathbb{R}^N, \quad (3.11)$$

for some $L = L(w) \in \mathbb{R}$. Multiplying by W , integrating, and summing up yields

$$\sum_{i=1}^k \left(\int_{\mathbb{R}^N} \nabla w_i \cdot \nabla W \, dx - \int_{\mathbb{R}^N} \frac{N(N+2)}{(1+|x|^2)^2} \sum_{j=1}^k a_{ij}(\bar{\alpha}) w_j(x) W(x) \, dx \right) = kL \int_{\mathbb{R}^N} |\nabla W|^2 \, dx.$$

Recalling that $-\Delta W = \frac{N(N+2)}{(1+|x|^2)^2} W$ (see Corollary 2.2) and $\sum_i a_{ij} = 1$, one sees that the left hand side of the equation vanishes and so $L = 0$. Thus, $w = (w_1, \dots, w_k)$ is a

solution to (1.9) and, using again Corollary 2.2 and assumptions (1.17)–(1.18), this is the case if and only if

$$w \in \text{span}\{(1, \dots, 1)W, \eta\}.$$

Recalling that $w \in K$, which means that w is orthogonal to $(1, \dots, 1)W$, and that $\eta \perp (1, \dots, 1)W$, one concludes that w is a multiple of η . \square

Lemma 3.7. *Under the assumptions of Lemma 3.6 the range $\text{Ran}(\partial_u T(\bar{\alpha}, U, \dots, U)) \subseteq K$ has codimension one. It is the set of functions $f = (f_1, \dots, f_k) \in K$ that are orthogonal to η in $(D^{1,2}(\mathbb{R}^N))^k$, that is*

$$(f|\eta) := \sum_{i=1}^k e_{n,i} \int_{\mathbb{R}^N} \nabla f_i \cdot \nabla W_n \, dx = 0 \quad (3.12)$$

where $e_n = (e_{n,i})_{i=1}^k$. Hence a complement of $\text{Ran}(\partial_u T(\bar{\alpha}, U, \dots, U))$ in K is spanned by the vector η defined in Lemma 3.6.

Proof. This is a consequence of Lemma 3.5. Indeed the operator $\partial_u T(\bar{\alpha}, U, \dots, U)$ can be written $\partial_u T(\bar{\alpha}, U, \dots, U)[w] = w - P_K((-\Delta)^{-1} \sum_{j=1}^k a_{ij}(\bar{\alpha}) \frac{N(N+2)}{(1+|x|^2)^2} w_j)_{i=1}^k$ because $w \in K$, and so is a compact perturbation of the identity. Thus (3.12) follows from the Fredholm Alternative. \square

Lemma 3.8. *Under the assumptions of Lemma 3.6 and (1.19), the operator T satisfies*

$$\partial_{\alpha u} T(\bar{\alpha}, U, \dots, U)[\eta] \notin \text{Ran}(\partial_u T(\bar{\alpha}, U, \dots, U)) \quad (3.13)$$

where η is as defined in (3.9).

Proof. The derivative $\partial_u T(\alpha, U, \dots, U)$ is given by (3.10). Differentiating with respect to α yields

$$\partial_{\alpha u} T(\bar{\alpha}, U, \dots, U)[\eta] = P_K g$$

where $g = (g_1, \dots, g_k)$ and

$$g_i := -(-\Delta)^{-1} \left(\sum_{j=1}^k \partial_{\alpha} a_{ij}(\bar{\alpha}) \frac{N(N+2)}{(1+|x|^2)^2} \eta_j \right), \quad i = 1, \dots, k.$$

In view of Lemma 3.7, we have to show that $(P_K g|\eta) \neq 0$. Since $\eta \in K$, $(P_K g|\eta) = (g|\eta)$. Thus, we have to show that

$$(g|\eta) = \sum_{i=1}^k e_{n,i} \int_{\mathbb{R}^N} \nabla \left(-(-\Delta)^{-1} \left(\sum_{j=1}^k \partial_{\alpha} a_{ij}(\bar{\alpha}) \frac{N(N+2)}{(1+|x|^2)^2} \eta_j \right) \right) \cdot \nabla W_n \, dx \neq 0,$$

that is, recalling that $\eta_j = e_{n,j} W_n$,

$$e_n \cdot \frac{dA}{d\alpha}(\bar{\alpha}) e_n \int_{\mathbb{R}^N} \frac{N(N+2)}{(1+|x|^2)^2} W_n^2(x) \, dx \neq 0.$$

The proof is complete thanks to assumption (1.19) (see also (1.20)). \square

Now we are in position to apply the bifurcation result of [CR]:

Proposition 3.9. *Assume that A satisfies (1.2) and (1.3). Assume further that there exists $\bar{\alpha}$ and $\bar{\imath}$ such that (1.17)–(1.19) are satisfied. Then the point $(\bar{\alpha}, U, \dots, U)$ is a radial bifurcation point for the curve $\alpha \mapsto (\alpha, U, \dots, U)$ of solutions to (1.1). More precisely, there exists continuous curves $\varepsilon \mapsto \alpha_\varepsilon$ and $\varepsilon \mapsto (u_{1,\varepsilon}, \dots, u_{k,\varepsilon})$, defined for $\varepsilon \in \mathbb{R}$ small enough, such that $\alpha_0 = \bar{\alpha}$, $u_{i,0} = U$, and*

$$\begin{cases} -\Delta u_{i,\varepsilon} = \sum_{j=1}^k a_{ij}(\alpha_\varepsilon) u_{j,\varepsilon}^{2^*-1} + L_\varepsilon \frac{N(N+2)}{(1+|x|^2)^2} W & \text{in } \mathbb{R}^N, \\ u_{i,\varepsilon} > 0, \quad u_{i,\varepsilon} \in D^{1,2}(\mathbb{R}^N), \end{cases} \quad (3.14)$$

for some Lagrange multiplier L_ε . Moreover, for ε small enough,

$$(u_{1,\varepsilon}, \dots, u_{k,\varepsilon}) = (1, \dots, 1)U + \varepsilon e_n W_n(|x|) + \varepsilon \varphi_\varepsilon(|x|) \quad (3.15)$$

where e_n is an eigenvector of A for the eigenvalue $\Lambda_{\bar{\imath}}(\bar{\alpha}) = \lambda_n$ and φ_ε is an uniformly bounded function in $(D^{1,2}(\mathbb{R}^N))^k$ and such that $\varphi_0 = 0$.

Proof. We apply Theorem 1.7 in [CR] at the operator T defined in (3.2). It is easy to see that $T(\alpha, U, \dots, U) = 0$ for any α . By Lemma 3.3 the operators $\partial_\alpha T$, $\partial_u T$ and $\partial_{\alpha,u} T$ are well defined and continuous from $\mathbb{R} \times (K \cap B^k)$ to K . Lemma 3.6 says that the kernel of $\partial_u T(\bar{\alpha}, U, \dots, U)$ is one-dimensional while Lemma 3.7 implies that its range has codimension one. Finally, Lemma 3.8 guarantees that the transversality condition holds. Therefore all assumptions of Theorem 1.7 in [CR] are satisfied. As a consequence, there exists a neighborhood V of $(\bar{\alpha}, U, \dots, U)$ in $\mathbb{R} \times (K \cap B^k)$, an interval $(-\varepsilon_0, \varepsilon_0)$, and continuous functions $(-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R} : \varepsilon \mapsto \alpha_\varepsilon$ and $(-\varepsilon_0, \varepsilon_0) \rightarrow B : \varepsilon \mapsto \varphi_{i,\varepsilon}$ for $i = 1, \dots, k$ such that $\alpha_0 = \bar{\alpha}$, $\varphi_{i,0} = 0$ for $i = 1, \dots, k$ and

$$T^{-1}(\{0\}) \cap V = \{(\alpha, U, \dots, U) \mid (\alpha, U, \dots, U) \in V\} \cup \{(\alpha_\varepsilon, u_{1,\varepsilon}, \dots, u_{k,\varepsilon}) \mid |\varepsilon| < \varepsilon_0\}$$

where $(u_{1,\varepsilon}, \dots, u_{k,\varepsilon})$ is defined by (3.15). In particular $T(\alpha_\varepsilon, u_{1,\varepsilon}, \dots, u_{k,\varepsilon}) = 0$ which means that $(u_{1,\varepsilon}, \dots, u_{k,\varepsilon})$ solves (3.14). This concludes the proof. \square

Lemma 3.10. *Let L_ε be the Lagrange multiplier of Proposition 3.9. Then*

$$|L_\varepsilon| \leq C. \quad (3.16)$$

Proof. Let us use the function W as test function in the first equation to (3.14). We get

$$N(N+2)L_\varepsilon \int_{\mathbb{R}^N} \frac{W^2}{(1+|x|^2)^2} dx = \int_{\mathbb{R}^N} \nabla u_{1,\varepsilon} \cdot \nabla W dx - \sum_{s=1}^k a_{1s}(\alpha) \int_{\mathbb{R}^N} u_{s,\varepsilon}^{2^*-1} W(x) dx.$$

The fact that $u_{s,\varepsilon}$ are uniformly bounded in $D^{1,2}(\mathbb{R}^N)$ then implies the claim. \square

3.3. The Pohozaev identity.

Proposition 3.11. *Suppose that $(u_i)_{i=1,\dots,k}$ are positive solutions in $D^{1,2}(\mathbb{R}^N)$ to*

$$-\Delta u_i = \sum_{j=1}^k a_{ij} u_j^{2^*-1} + H_i(x) \quad \text{in } \mathbb{R}^N \quad (3.17)$$

where H_i are smooth functions satisfying

$$H_i \in L^{2^*}(\mathbb{R}^N), \quad |x|H_i \in L^2(\mathbb{R}^N) \quad (3.18)$$

and A is an invertible symmetric matrix. Let us write $A^{-1} = (a_{ij}^{-1})_{i,j=1,\dots,k}$. Then, the following Pohozaev identity holds

$$0 = \sum_{i,j=1}^k a_{ij}^{-1} \int_{\mathbb{R}^N} H_i(x) \left(x \cdot \nabla u_j + \frac{N-2}{2} u_j \right) dx \quad (3.19)$$

Proof. We will denote by $I_{i,R}$ various boundary terms on ∂B_R such that, for any integer i ,

$$|I_{i,R}| \leq C(N)R \int_{\partial B_R} \left(\sum_{h=1}^k u_i u_h^{2^*-1} + |\nabla u_i \cdot \nabla u_h| \right). \quad (3.20)$$

Set $\sum_{h=1}^k u_i u_h^{2^*-1} + |\nabla u_i \cdot \nabla u_h| =: G(u_1, \dots, u_k)$ and, as in [BL], let us show that there exists a sequence $R_n \rightarrow +\infty$ such that $I_{i,R_n} \rightarrow 0$. Indeed since $u_i \in D^{1,2}(\mathbb{R}^N)$ we know that $G(u_1, \dots, u_k) \in L^1(\mathbb{R}^N)$ so that

$$\int_0^{+\infty} \int_{\partial B_R} G(u_1, \dots, u_k) d\sigma dR < +\infty.$$

Hence, there exists a sequence $R_n \rightarrow +\infty$ such that

$$R_n \int_{\partial B_{R_n}} G(u_1, \dots, u_k) d\sigma \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad (3.21)$$

and this shows that $I_{i,R_n} \rightarrow 0$. From now, to simplify the notations, we agree that $R = R_n$ and we denote by $C(R)$ a linear combination of $I_{i,R}$.

Let us now start our main argument with the identity:

$$-\int_{B_R} \Delta u_i (x \cdot \nabla u_i) = \left(1 - \frac{N}{2}\right) \int_{B_R} |\nabla u_i|^2 dx - \underbrace{\int_{\partial B_R} (x \cdot \nabla u_i) \frac{\partial u_i}{\partial \nu}}_{=I_{1,R}} + \frac{1}{2} \underbrace{\int_{\partial B_R} |\nabla u_i|^2 (x \cdot \nu)}_{=I_{2,R}}$$

Using the i -th equation in (3.17), we get

$$\left(1 - \frac{N}{2}\right) \int_{B_R} |\nabla u_i|^2 dx + C(R) = \sum_{j=1}^k a_{ij} \int_{B_R} u_j^{\frac{N+2}{N-2}} (x \cdot \nabla u_i) dx + \int_{B_R} H_i(x) (x \cdot \nabla u_i) dx.$$

Next we estimate

$$\int_{B_R} u_j^{\frac{N+2}{N-2}} (x \cdot \nabla u_i) = -N \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} - \frac{N+2}{N-2} \int_{B_R} u_i u_j^{\frac{4}{N-2}} (x \cdot \nabla u_j) + \underbrace{\int_{\partial B_R} u_i u_j^{\frac{N+2}{N-2}} (x \cdot \nu)}_{=I_{3,R}} \quad (3.22)$$

which, when $i = j$, simplifies to

$$\int_{B_R} u_i^{\frac{N+2}{N-2}} (x \cdot \nabla u_i) = -\frac{N-2}{2} \int_{B_R} u_i^{\frac{2N}{N-2}} + \underbrace{\frac{N-2}{2N} R \int_{B_R} u_i^{\frac{2N}{N-2}}}_{=I_{4,R}} \quad (3.23)$$

Using (3.22) and (3.23) we get

$$\begin{aligned} \left(1 - \frac{N}{2}\right) \int_{B_R} |\nabla u_i|^2 dx + C(R) &= -a_{ii} \frac{N-2}{2} \int_{B_R} u_i^{\frac{2N}{N-2}} - N \sum_{j \neq i} a_{ij} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} \\ &\quad - \frac{N+2}{N-2} \sum_{j \neq i} a_{ij} \int_{B_R} (x \cdot \nabla u_j) u_i u_j^{\frac{4}{N-2}} + \int_{B_R} H_i(x) (x \cdot \nabla u_i) dx. \end{aligned} \quad (3.24)$$

On the other hand, multiplying equation (3.17) by u_i and integrating yields

$$\int_{B_R} |\nabla u_i|^2 + C(R) = a_{ii} \int_{B_R} u_i^{\frac{2N}{N-2}} + \sum_{j \neq i} a_{ij} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} + \int_{B_R} H_i(x) u_i dx. \quad (3.25)$$

Summing (3.24) and (3.25) multiplied by $\frac{N-2}{2}$ gives

$$\begin{aligned} \frac{N+2}{2} \sum_{j \neq i} a_{ij} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} + \frac{N+2}{N-2} \sum_{j \neq i} a_{ij} \int_{B_R} (x \cdot \nabla u_j) u_i u_j^{\frac{4}{N-2}} + C(R) \\ = \int_{B_R} H_i(x) \left(x \cdot \nabla u_i + \frac{N-2}{2} u_i \right) \end{aligned} \quad (3.26)$$

Setting

$$A_{ij} := \frac{N+2}{2} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} + \frac{N+2}{N-2} \int_{B_R} (x \cdot \nabla u_j) u_i u_j^{\frac{4}{N-2}}$$

and

$$B_{ij} := \int_{B_R} H_i(x) \left(x \cdot \nabla u_j + \frac{N-2}{2} u_j \right), \quad (3.27)$$

the previous identity becomes

$$\boxed{\sum_{j \neq i} a_{ij} A_{ij} = B_{ii} + C(R).} \quad (3.28)$$

Now let us use the factor $x \cdot \nabla u_h$ against u_i . Let us start with the identity:

$$-\int_{B_R} \Delta u_i (x \cdot \nabla u_h) = \int_{B_R} \nabla u_i \cdot \nabla u_h + \sum_{\ell, m=1}^N \int_{B_R} x_\ell \frac{\partial u_i}{\partial x_m} \frac{\partial^2 u_h}{\partial x_\ell \partial x_m} - \underbrace{\int_{B_R} \frac{\partial u_i}{\partial \nu} (x \cdot \nabla u_h)}_{=I_{5,R}}$$

Using (3.17), one gets

$$\begin{aligned}
& \int_{B_R} \nabla u_i \cdot \nabla u_h + \sum_{\ell, m=1}^N \int_{B_R} x_\ell \frac{\partial u_i}{\partial x_m} \frac{\partial^2 u_h}{\partial x_\ell \partial x_m} + C(R) \\
&= -N \sum_{j \neq h} a_{ij} \int_{B_R} u_h u_j^{\frac{N+2}{N-2}} - \frac{N+2}{N-2} \sum_{j \neq h} a_{ij} \int_{B_R} (x \cdot \nabla u_j) u_h u_j^{\frac{4}{N-2}} - a_{ih} \frac{N-2}{2} \int_{B_R} u_h^{\frac{2N}{N-2}} \\
&+ \int_{B_R} H_i(x) (x \cdot \nabla u_h) dx. \tag{3.29}
\end{aligned}$$

Our intention is to sum (3.29) and the same expression with the indices i and h swapped. Let us start by remarking that

$$\sum_{\ell, m=1}^N \int_{B_R} x_\ell \left(\frac{\partial u_i}{\partial x_m} \frac{\partial^2 u_h}{\partial x_\ell \partial x_m} + \frac{\partial u_h}{\partial x_m} \frac{\partial^2 u_i}{\partial x_\ell \partial x_m} \right) = -N \int_{B_R} \nabla u_i \cdot \nabla u_h + R \underbrace{\int_{B_R} \nabla u_i \cdot \nabla u_h}_{=I_{6,R}}. \tag{3.30}$$

Using (3.30), the sum of (3.29) and its symmetric expression reads

$$\begin{aligned}
& (2-N) \int_{B_R} \nabla u_i \cdot \nabla u_h + C(R) \\
&= -N \sum_{j \neq h} a_{ij} \int_{B_R} u_h u_j^{\frac{N+2}{N-2}} - \frac{N+2}{N-2} \sum_{j \neq h} a_{ij} \int_{B_R} (x \cdot \nabla u_j) u_h u_j^{\frac{4}{N-2}} - a_{ih} \frac{N-2}{2} \int_{B_R} u_h^{\frac{2N}{N-2}} \\
&- N \sum_{j \neq i} a_{hj} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} - \frac{N+2}{N-2} \sum_{j \neq i} a_{hj} \int_{B_R} (x \cdot \nabla u_j) u_i u_j^{\frac{4}{N-2}} - a_{hi} \frac{N-2}{2} \int_{B_R} u_i^{\frac{2N}{N-2}} \\
&+ \int_{B_R} H_i(x) (x \cdot \nabla u_h) + \int_{B_R} H_h(x) (x \cdot \nabla u_i). \tag{3.31}
\end{aligned}$$

Using again the i -th equation of (3.17) but this time multiplying by u_h yields

$$\int_{B_R} \nabla u_i \cdot \nabla u_h + C(R) = \sum_{j \neq h} a_{ij} \int_{B_R} u_h u_j^{\frac{N+2}{N-2}} + a_{ih} \int_{B_R} u_h^{\frac{2N}{N-2}} + \int_{B_R} H_i(x) u_h dx. \tag{3.32}$$

Now, let us write $2 \int_{B_R} \nabla u_i \cdot \nabla u_h = \int_{B_R} \nabla u_i \cdot \nabla u_h + \int_{B_R} \nabla u_h \cdot \nabla u_i$ and substitute the first term using (3.32) and the second term using (3.32) with i and h swapped. Let us then multiply the resulting expression by $\frac{N-2}{2}$ and add it to (3.31). This gives the

following equality:

$$\begin{aligned}
& \frac{N+2}{2} \sum_{j \neq h} a_{ij} \int_{B_R} u_h u_j^{\frac{N+2}{N-2}} + \frac{N+2}{N-2} \sum_{j \neq h} a_{ij} \int_{B_R} (x \cdot \nabla u_j) u_h u_j^{\frac{4}{N-2}} \\
& + \frac{N+2}{2} \sum_{j \neq i} a_{hj} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} + \frac{N+2}{N-2} \sum_{j \neq i} a_{hj} \int_{B_R} (x \cdot \nabla u_j) u_i u_j^{\frac{4}{N-2}} \\
& = \int_{B_R} H_i(x) \left(x \cdot \nabla u_h + \frac{N-2}{2} u_h \right) + \int_{B_R} H_h(x) \left(x \cdot \nabla u_i + \frac{N-2}{2} u_i \right) + C(R)
\end{aligned} \tag{3.33}$$

Recalling the definition of A_{ij} and B_{ij} , one can write (3.33) as

$$\boxed{\sum_{j \neq h} a_{ij} A_{hj} + \sum_{j \neq i} a_{hj} A_{ij} = B_{ih} + B_{hi} + C(R).} \tag{3.34}$$

Now let us multiply (3.28) by a_{ii}^{-1} for $i = 1, \dots, k$ and sum on i . We get that (3.28) becomes

$$\sum_{i=1}^k \sum_{j \neq i} a_{ii}^{-1} a_{ij} A_{ij} = \sum_{i=1}^k a_{ii}^{-1} B_{ii} + C(R).$$

Multiplying (3.34) by a_{ih}^{-1} and summing on the triangular bloc of the indices (i, h) satisfying $1 \leq i < h \leq k$, we get

$$\sum_{i=1}^{k-1} \sum_{h=i+1}^k a_{ih}^{-1} \left(\sum_{j \neq h} a_{ij} A_{hj} + \sum_{j \neq i} a_{hj} A_{ij} \right) = \sum_{i=1}^{k-1} \sum_{h=i+1}^k a_{ih}^{-1} (B_{ih} + B_{hi}) + C(R).$$

Finally summing up the previous two relations, we get

$$\begin{aligned}
& \sum_{i=1}^{k-1} \sum_{h=i+1}^k a_{ih}^{-1} \left(\sum_{j \neq h} a_{ij} A_{hj} + \sum_{j \neq i} a_{hj} A_{ij} \right) + \sum_{i=1}^k \sum_{j \neq i} a_{ii}^{-1} a_{ij} A_{ij} \\
& = \sum_{i=1}^{k-1} \sum_{h=i+1}^k a_{ih}^{-1} (B_{ih} + B_{hi}) + \sum_{i=1}^k a_{ii}^{-1} B_{ii} + C(R). \tag{3.35}
\end{aligned}$$

Let us consider the RHS of (3.35) and observe that, using the symmetry of the matrix A^{-1} , we have

$$\begin{aligned}
& \sum_{i=1}^{k-1} \sum_{h=i+1}^k a_{ih}^{-1} (B_{ih} + B_{hi}) + \sum_{i=1}^k a_{ii}^{-1} B_{ii} \\
& = \sum_{i=1}^{k-1} \sum_{h=i+1}^k a_{ih}^{-1} B_{ih} + \sum_{h=1}^{k-1} \sum_{i=h+1}^k a_{ih}^{-1} B_{ih} + \sum_{i=1}^k a_{ii}^{-1} B_{ii} = \sum_{i=1}^k \sum_{h=1}^k a_{ih}^{-1} B_{ih}
\end{aligned}$$

where the last equality results from the fact that all $(i, h) \in \{1, \dots, k\}^2$ are present in the previous terms: all $i < h$ in the first double sum, $i > h$ in the second one, and $i = h$ in the third sum.

Doing the same kind of computation for the LHS, we have

$$\begin{aligned} \sum_{i=1}^{k-1} \sum_{h=i+1}^k a_{ih}^{-1} \left(\sum_{j \neq h} a_{ij} A_{hj} + \sum_{j \neq i} a_{hj}^{-1} A_{ij} \right) + \sum_{i=1}^k \sum_{j \neq i} a_{ii}^{-1} a_{ij} A_{ij} \\ = \sum_{i=1}^k \sum_{j \neq i} A_{ij} \left(\sum_{h=1}^k a_{ih}^{-1} a_{hj} \right) = \sum_{i=1}^k \sum_{j \neq i} A_{ij} \delta_i^j = 0 \end{aligned}$$

Therefore, (3.35) reads

$$\sum_{i=1}^k \sum_{h=1}^k a_{ih}^{-1} B_{ih} + C(R) = 0.$$

From the summability assumptions on u_i and H , we can pass to the limit along the sequence $R_n \rightarrow +\infty$ chosen at the beginning of this proof and get

$$\sum_{i=1}^k \sum_{h=1}^k a_{ih}^{-1} \int_{\mathbb{R}^N} H_i(x) \left(x \cdot \nabla u_h + \frac{N-2}{2} u_h \right) = 0. \quad \square$$

Lemma 3.12. *Let A be an invertible matrix satisfying (1.2), (1.3) and denote a_{ij}^{-1} the entries of A^{-1} . Then*

$$\sum_{i,j=1}^k a_{ij}^{-1} = k.$$

Proof. Assumption (1.3) can be written $A(1, \dots, 1) = (1, \dots, 1)$. Multiplying both sides by $(1, \dots, 1)^\top A^{-1}$, one gets

$$\sum_{i,j=1}^k a_{ij}^{-1} = (1, \dots, 1)^\top A^{-1} (1, \dots, 1) = (1, \dots, 1)^\top (1, \dots, 1) = k. \quad \square$$

We are in position to prove our main result:

Proof of Theorem 1.3. Proposition 3.9 says that there exist $u_{j,\varepsilon}$ satisfying (3.14) for ε small enough. Assumption (1.21) says that the matrix A is invertible at $\bar{\alpha}$ and then from Proposition 3.11, we get

$$L_\varepsilon \sum_{i,j=1}^k a_{ij}^{-1} \int_{\mathbb{R}^N} \frac{N(N+2)}{(1+|x|^2)^2} W(x) \left(x \cdot \nabla u_{i,\varepsilon} + \frac{N-2}{2} u_{i,\varepsilon} \right) dx = 0. \quad (3.36)$$

Recalling that $u_{i,\varepsilon} \rightarrow U$ in $D^{1,2}(\mathbb{R}^N)$ when $\varepsilon \rightarrow 0$, one can pass to the limit and get

$$\int_{\mathbb{R}^N} \frac{N(N+2)}{(1+|x|^2)^2} W(x) \left(x \cdot \nabla u_{i,\varepsilon} + \frac{N-2}{2} u_{i,\varepsilon} \right) dx \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{N(N+2)}{(1+|x|^2)^2} W^2(x) dx \neq 0.$$

Thanks to Lemma 3.12, $\sum_{i,j=1}^k a_{ij}^{-1} \neq 0$ and then (3.36) implies that $L_\varepsilon = 0$ for ε small enough, concluding the proof. \square

APPENDIX A. COMPUTATION OF THE FIRST DERIVATIVE OF THE PARAMETER

In this appendix we give some information on the behavior of branch of solutions of Theorem 1.3. Let us recall that the bifurcation is called transcritical if

$$\left. \frac{d\alpha_\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} \neq 0. \quad (\text{A.1})$$

Although in the literature are present formulas for the calculation of the derivative of α_ε (see for example [K]), it seems difficult to provide a complete characterization of the bifurcation diagram. In next proposition we give a sufficient condition to have a transcritical bifurcation.

Proposition A.1. *Let us suppose that*

$$\sum_{j=1}^k e_{n,j}^3 \int_{\mathbb{R}^N} U^{\frac{6-N}{N-2}} W_n^3 dx \neq 0, \quad (\text{A.2})$$

(see (3.9) for the definition of e_n). Then the bifurcation given in Theorem 1.3 is transcritical.

Remark A.2. If $k = 2$ we have $e_n = (1, -1)$ and so (A.2) is never satisfied. In this case we need refined estimates involving higher order derivatives. We do not investigate this situation. On the other hand, if $k \geq 3$ it is easy to find matrices A verifying $\sum_{j=1}^k e_{n,j}^3 \neq 0$. Finally, in the special case $N = 4$ and $n = 2$ we get

$$\int_{\mathbb{R}^N} U^{\frac{6-N}{N-2}} W_n^3 dx = \int_{\mathbb{R}^4} U W_2^3 dx = \int_{-1}^1 (1-\xi^2) (P_2^{(1,1)}(\xi))^3 d\xi = \frac{27}{64} \int_{-1}^1 (1-\xi^2) (5\xi^2-1)^3 d\xi \neq 0. \quad (\text{A.3})$$

Proof. Using the formula (1.6.3) at page 21 in [K] we get,

$$\left. \frac{d\alpha_\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} = -\frac{1}{2} \frac{(\partial_u^2 T(\bar{\alpha}, U, \dots, U)[\eta, \eta] \mid \eta)}{(\partial_{\alpha u} T(\bar{\alpha}, U, \dots, U)[\eta] \mid \eta)} \quad (\text{A.4})$$

where η is defined in (3.9) and $(\cdot \mid \cdot)$ denotes the inner product in $(D^{1,2}(\mathbb{R}^N))^k$. Let us compute the numerator. Given the definition (3.2) of T , one easily gets

$$\partial_u^2 T(\alpha, U, \dots, U)[w, w'] = P_K \left(-(-\Delta)^{-1} \left(\sum_{j=1}^k a_{ij}(\alpha) U_2 w_j w'_j \right) \right)_{i=1}^k$$

where $U_2 := \frac{4(N+2)}{(N-2)^2} U^{\frac{6-N}{N-2}}$. In particular, in view of the definition of η (see (3.9)),

$$\partial_u^2 T(\bar{\alpha}, U, \dots, U)[\eta, \eta] = P_K \left(-(-\Delta)^{-1} \left(\sum_{j=1}^k a_{ij}(\bar{\alpha}) e_{n,j}^2 U_2 W_n^2 \right) \right)_{i=1}^k$$

Since $\eta \in K$, we can drop P_K when performing the inner product. Thus the numerator reads:

$$\begin{aligned} \sum_{i=1}^k e_{n,i} \int_{\mathbb{R}^N} \nabla \left(-(-\Delta)^{-1} \left(\sum_{j=1}^k a_{ij}(\bar{\alpha}) e_{n,j}^2 U_2 W_n^2 \right) \right) \cdot \nabla W_n \, dx \\ = -e_{n,i} a_{ij}(\bar{\alpha}) e_{n,j}^2 \int_{\mathbb{R}^N} U_2 W_n^3 \, dx \quad (\text{A.5}) \end{aligned}$$

Recalling that e_n is an eigenvector of $A(\bar{\alpha})$ for the eigenvalue $\Lambda_i(\bar{\alpha})$, one gets from (A.5)

$$\left. \frac{d\alpha_\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} = 0 \Leftrightarrow -\Lambda_i(\bar{\alpha}) \sum_{j=1}^k e_{n,j}^3 \int_{\mathbb{R}^N} U_2 W_n^3 \, dx = 0$$

which gives the claim. \square

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FRANCESCA GLADIALI, DIPARTIMENTO POLCOMING, UNIVERSITÀ DI SASSARI - VIA PIANDANNA 4, 07100 SASSARI, ITALY.

E-mail address: `fgladiali@uniss.it`

MASSIMO GROSSI, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA LA SAPIENZA, P.LE A. MORO 2 - 00185 ROMA, ITALY.

E-mail address: `massimo.grossi@uniroma1.it`

CHRISTOPHE TROESTLER, DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ DE MONS, PLACE DU PARC 20, B-7000 MONS, BELGIUM.

E-mail address: `Christophe.Troestler@umons.ac.be`